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Capacity and Absolute Abel Summability of Multiple Fourier Series

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1. INTRODUCTION

Operating in Euclidean N-space, $N \geqslant 2$, we shall use the following notation: $x = (x_1, \ldots, x_N)$, $(x, y) = x_1 y_1 + \cdots + x_N y_N$, $|x| = (x, x)^{1/2}$, and $\alpha x + \beta y = (\alpha x_1 + \beta y_1, \ldots, \alpha x_N + \beta y_N)$. $B(x, r)$, $r > 0$, will designate the open N-ball with center x and radius r and $T_N = \{x: -\pi \leqslant x_j < \pi, j = 1, \ldots, N\}$ will designate the N-dimensional torus.

For f in $L^1(T_N)$ and m an integral lattice point, we shall set

$$f^\wedge(m) = (2\pi)^{-N} \int_{T_N} e^{-i(m,x)} f(x)\, dx.$$

$\mathscr{M}(T_N)$ will designate the class of finite Borel measures on T_N, and, for μ in $\mathscr{M}(T_N)$, we shall set

$$\mu^\wedge(m) = (2\pi)^{-N} \int_{T_N} e^{-i(m,x)}\, d\mu(x).$$

For $t > 0$, we introduce the function

$$H(x, t) = \sum_{m \neq 0} e^{i(m,x) - |m| t} / |m|^2 \tag{1.1}$$

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and observe from [7, Lemma 8] that

$$\lim_{t\to 0} H(x, t) = H(x) \text{ exists and is finite for } x \text{ in } E_N - \bigcup_m \{2\pi m\};$$
$$\lim_{t\to 0} H(2\pi m, t) = H(2\pi m) = +\infty. \tag{1.2}$$

We also observe from this reference that the following hold:

$$H(x) - |x|^2/2N \text{ is harmonic in } E_N - \bigcup_m \{2\pi m\}; \tag{1.3}$$

there are positive finite constants b_N and b_N' such that

$$\sup_{x \text{ in } T_N - 0} |H(x) - b_N |x|^{2-N}| \leqslant b_N' \quad \text{for} \quad N \geqslant 3,$$
$$\sup_{x \text{ in } T_N - 0} |H(x) - b_N \log(1/|x|)| \leqslant b_N' \quad \text{for} \quad N = 2; \tag{1.4}$$

$H(x)$ is in $L^1(T_N)$,

$$H^\wedge(m) = |m|^{-2} \quad \text{for} \quad m \neq 0, \quad \text{and} \quad H^\wedge(0) = 0. \tag{1.5}$$

Let $Z \subset T_N$ be a set closed in the torus topology. Following the classical concepts concerning capacity theory and in view of (1.3) and (1.4), we shall say Z is of ordinary capacity zero if

$$\int_{T_N} \int_{T_N} H(x - y)\, d\mu(x)\, d\mu(y) = +\infty$$

for every nonnegative μ in $\mathscr{M}(T_N)$ of total mass one having its support contained in Z, i.e., $\mu(Z) = 1$ and $\mu(T_N - Z) = 0$.

In view of [3, pp. 3 and 24], we shall say an analytic set $A \subset T_N$ is of ordinary capacity zero if every set $Z \subset A$ which is closed in the torus topology is of ordinary capacity zero.

Next for f in $L^1(T_N)$, we shall set

$$f(x, t) = \sum_m f^\wedge(m)\, e^{i(m,x) - |m|t} \quad \text{for} \quad t > 0 \tag{1.6}$$

and refer to $f(x, t)$ as the Abelian means of f. We observe that for fixed x^0, $f(x^0, t)$ is in $C^\infty(0, \infty)$ as a function of t. Consequently, following the classical terminology (see [8, p. 83]) we shall say the Fourier series of f is absolutely Abel summable at x^0 if

$$\int_0^1 |\partial f(x^0, t)/\partial t|\, dt < \infty. \tag{1.7}$$

It is easy to see that if (1.7) holds, then $\lim_{t\to 0} f(x^0, t)$ exists and is finite.

Motivated by the one-dimensional work of Beurling (see [5, pp. 47 and 49] and [1]), we intend to establish the following two theorems connecting absolute Abel summability with ordinary capacity.

THEOREM 1. *Let $Z \subset T_N$ be closed in the torus topology, and let f be in $L^1(T_N)$. Suppose that*

(i) $\sum_m | m |^2 | f^\wedge(m)|^2 < +\infty$

and

(ii) $\int_0^1 | \partial f(x, t)/\partial t | \, dt = +\infty$ *for x in Z.*

Then Z is of ordinary capacity zero.

THEOREM 2. *Let $Z \subset T_N$ be closed in the torus topology and suppose that Z is of ordinary capacity zero. Then there exists an f in $L^1(T_N)$ with $\sum_m | m |^2 | f^\wedge(m)|^2 < +\infty$ such that*

$$\int_0^1 | \partial f(x, t)/\partial t | \, dt = +\infty \qquad \text{for } x \text{ in } Z.$$

(For Theorem 2, see also [6].)

It is easy to see that for f in $L^1(T_N)$, the set $\{x : x \text{ in } T_N \text{ and } \int_0^1 | \partial f(x, t)/\partial t | \, dt = +\infty\}$ is a G_δ-set in the torus topology. Consequently, we obtain as an immediate corollary to Theorem 1, the following corollary.

COROLLARY 1. *Let f be in $L^1(T_N)$ and suppose that $\sum_m | m |^2 | f^\wedge(m)|^2 < \infty$. Then the Fourier series of f is absolutely Abel summable except possibly on a set of ordinary capacity zero.*

It is also easy to show using Theorem 1 that *under the same hypothesis as Corollary* 1, *the Fourier series of f is spherically convergent except possibly on a set of ordinary capacity zero.* We leave the proof of this fact to the interested reader.

2. PROOF OF THEOREM 1

In order to prove Theorem 1, we set

$$G(x, t) = \sum_{m \neq 0} e^{i(m,x) - |m| t} / | m | \qquad \text{for} \quad t > 0, \tag{2.1}$$

and establish the following facts:

$$\lim_{t \to 0} G(x, t) = G(x) \text{ exists in } E_N ; \tag{2.2}$$

$G(x)$ is continuous in

$$E_N - \bigcup_m \{2\pi m\} \qquad \text{and} \qquad G(2\pi m) = +\infty; \tag{2.3}$$

$G(x)$ is in $L^1(T_N)$, and for $t > 0$, $G(x, t)$ is the Abelian means of G; (2.4)

$G(x) + \alpha_N \geqslant 0$ for x in E_N where

$$\alpha_N = 2 + \max_{x \text{ in } T_n} |G(x, 1)|; \tag{2.5}$$

$$\int_0^1 |\partial G(x, t)/\partial t| \, dt \leqslant G(x) + \alpha_N \qquad \text{for } x \text{ in } E_N. \tag{2.6}$$

In the sequel, we shall also use the notation $\partial G(x, t)/\partial t = G_t(x, t)$.

In order to establish (2.2), we first observe from (2.1) and [2, p. 32] that for $t > 0$

$$\begin{aligned} -G_t(x, t) &= \sum_{m \neq 0} e^{i(m,x)-|m|t} \\ &= \gamma_N \sum_m t\{t^2 + |2\pi m + x|^2\}^{-(N+1)/2} - 1. \end{aligned} \tag{2.7}$$

where γ_N is a positive constant.

Given x^0 in $E_N - \bigcup_m \{2\pi m\}$, we see from (2.7) that there exists $h_0 > 0$ such that $|G_t(x, t)|$ is uniformly bounded for x in $B(x^0, h_0)$ and $0 < t < 1$. Consequently, $G(x, t)$ satisfies a uniform Cauchy criterion as $t \to 0$ for x in $B(x^0, h_0)$, and both (2.2) and (2.3) are established for x in $E_N - \bigcup_m \{2\pi m\}$. Obviously, $\lim_{t\to 0} G(2\pi m, t) = +\infty$; so (2.2) and (2.3) are entirely established.

To establish (2.4), we observe from (2.7) that there is a constant K such that

$$|G_t(x, t) + \gamma_N t(t^2 + |x|^2)^{-(N+1)/2}| \leqslant K \tag{2.8}$$

for x in $T_N - 0$ and $0 < t \leqslant 1$.

Observing that $G(x, 1)$ is a continuous periodic function in E_N, we conclude from (2.8) that there is a constant K' such that

$$|G(x, t) - \gamma_N(N-1)^{-1}(t^2 + |x|^2)^{(1-N)/2}| \leqslant K' \tag{2.9}$$

for x in $T_N - 0$ and $0 < t < 1$.

From (2.9), we obtain that

$$|G(x, t)| \leqslant K' + \gamma_N(N-1)^{-1} |x|^{1-N} \tag{2.10}$$

for x in $T_N - 0$ and $0 < t \leqslant 1$.

Observing first that the expression on the right side of the inequality in (2.10) is $L^1(T_N)$ and next from (2.2) that $G(x, t) \to G(x)$ for x in $T_N - 0$ as $t \to 0$, we conclude from (2.10) that

$$G(x) \text{ is } L^1(T_N) \tag{2.11}$$

and, furthermore, that

$$\int_{T_N} | G(x, t) - G(x)| \, dx \to 0 \quad \text{as} \quad t \to 0. \tag{2.12}$$

Equation (2.11) is the same as the first part of (2.4). Also (2.1), (2.11), and (2.12) imply that $G\hat{}(m) = | m |^{-1}$ for $m \neq 0$ and $G\hat{}(0) = 0$. This gives us the second part of (2.4), and (2.4) is completely established.

To establish (2.5), we observe from (2.7) that for x in E_N and $0 < t < 1$

$$G(x, t) - G(x, 1) + (1 - t) = \gamma_N (N - 1)^{-1} \sum_m \{(t^2 + | 2\pi m + x |^2)^{(1-N)/2} - (1 + | 2\pi m + x |^2)^{(1-N)/2}\}. \tag{2.13}$$

We conclude from (2.13) that for x in E_N and $0 < t < 1$, $G(x, t) - G(x, 1) + (1 - t) \geqslant 0$. But then we have from (2.2) that

$$G(x) - G(x, 1) + 1 \geqslant 0 \quad \text{for} \quad x \text{ in } E_N . \tag{2.14}$$

Observing once again that $G(x, 1)$ is a continuous periodic function, we see that (2.5) follows immediately from (2.14).

To establish (2.6), we observe from (2.7) that for x in E_N and $t > 0$,

$$| G_t(x, t)| \leqslant 2 - G_t(x, t). \tag{2.15}$$

Consequently, we conclude from (2.2), (2.3), and (2.15) that

$$\int_0^1 | G_t(x, t)| \, dt \leqslant 2 + G(x) - G(x, 1) \tag{2.16}$$

for x in E_N . Equation (2.6) follows immediately from (2.16).

We are now ready to prove the theorem. Assume to the contrary that Z has positive ordinary capacity. Then it follows that there is a non-negative Borel measure μ in $\mathscr{M}(T_N)$ such that

$$\mu(Z) = 1 \quad \text{and} \quad \mu(T_N - Z) = 0 \tag{2.17}$$

and, furthermore, such that

$$\int_Z \int_Z H(x - y) \, d\mu(x) \, d\mu(y) < +\infty. \tag{2.18}$$

Next, we observe from condition (i) in the hypothesis of the theorem that there exists F in $L^2(T_N)$ such that

$$F^{\wedge}(m) = |m| f^{\wedge}(m) \qquad \text{for every } m. \tag{2.19}$$

It follows from (1.6) that for $t > 0$,

$$f_t(x, t) = -\sum_m |m| f^{\wedge}(m) e^{i(m,x)-|m|t}. \tag{2.20}$$

From (2.7), (2.19), and (2.20), we conclude that for $t > 0$,

$$f_t(x, t) = (2\pi)^{-N} \int_{T_N} F(y) G_t(x - y, t) \, dy.$$

But then

$$\int_0^1 |f_t(x, t)| \, dt \leqslant (2\pi)^{-N} \int_{T_N} |F(y)| \left[\int_0^1 |G_t(x - y, t)| \, dt\right] dy.$$

From (2.5) and (2.6), we in turn obtain from this last fact that

$$\int_0^1 |f_t(x, t)| \, dt \leqslant (2\pi)^{-N} \int_{T_N} |F(y)| \, [G(x - y) + \alpha_N] \, dy,$$

and we conclude that

$$\int_Z \left[\int_0^1 |f_t(x, t)| \, dt\right] d\mu(x) \leqslant (2\pi)^{-N} \int_{T_N} |F(y)| \left\{\int_Z [G(x - y) + \alpha_N] \, d\mu(x)\right\} dy. \tag{2.21}$$

Next, we observe from (2.1)–(2.5), (2.7), and (2.9) that for x and z in T_N,

$$\int_{T_N} [G(x - y) + \alpha_N][G(z - y) + \alpha_N] \, dy = \lim_{t \to 0} \int_{T_N} [G(x - y, t) + \alpha_N][G(z - y, t) + \alpha_N] \, dy. \tag{2.22}$$

On the other hand, an easy computation using (2.1), (1.1), and (1.2) shows that the expression on the right side of the equality in (2.22) is equal to $(2\pi)^N [H(x - z) + \alpha_N^2]$.

We, consequently, conclude from this fact, (2.22), and Fubini's theorem that

$$\int_{T_N} \left\{\int_Z [G(x - y) + \alpha_N] \, d\mu(x)\right\}^2 dy = (2\pi)^N \int_Z \int_Z [H(x - z) + \alpha_N^2] \, d\mu(x) \, d\mu(z). \tag{2.23}$$

From (2.18) we have that the expression on the right side of (2.23) is finite. We, consequently, conclude from (2.23) that

$$\int_Z [G(x - y) + \alpha_N] \, d\mu(x) \text{ is in } L^2(T_N). \tag{2.24}$$

Next, we combine (2.19) with (2.24) and conclude from (2.21) and Schwartz's inequality that

$$\int_Z \left[\int_0^1 |f_t(x, t)| \, dt\right] d\mu(x) < +\infty. \tag{2.25}$$

On the other hand, we have from condition (i) in the hypothesis of the theorem and from (2.17) that

$$\int_Z \left[\int_0^1 |f_t(x, t)| \, dt\right] d\mu(x) = +\infty. \tag{2.26}$$

Equations (2.25) and (2.26) are mutually contradictory. Consequently, Z must be of ordinary capacity zero and the proof of the theorem is complete.

3. Proof of Theorem 2

With $H(x)$ defined as in (1.2), we see from (1.3) and (1.4) that there is a positive constant η_N such that

$$H(x) + \eta_N \geqslant 1 \qquad \text{for all} \quad x \text{ in } T_N . \tag{3.1}$$

We set

$$\Phi(x) = H(x) + \eta_N \tag{3.2}$$

and observe in particular that

$$\begin{aligned} \Phi\hat{\ }(0) &= \eta_N > 0, \\ \Phi\hat{\ }(m) &= |m|^{-2} \qquad \text{for} \quad m \neq 0. \end{aligned} \tag{3.3}$$

Next, we set

$$\Phi(x, t) = \sum_m \Phi\hat{\ }(m) \, e^{i(m,x) - |m|t} \tag{3.4}$$

and observe from (2.1) and (3.3) that for $t > 0$

$$\Phi_t(x, t) = -G(x, t). \tag{3.5}$$

From [7, p. 56 (17)], we see that if a periodic function is nonnegative on T_N, its Abelian means are also nonnegative on T_N. As a consequence of this fact, (2.5), (3.5) and the mean value theorem, we conclude that for $t > 0$ and x in T_N,

$$0 \leqslant \Phi(x, t) \leqslant \Phi(x) + \alpha_N t, \tag{3.6}$$

where α_N is the positive constant in (2.5).

It follows from (3.3), (3.4), (3.6), and Fatou's lemma that if μ is a nonnegative measure in $\mathscr{M}(T_N)$ then

$$\int_{T_N} \int_{T_N} \Phi(x - y)\, d\mu(x)\, d\mu(y) = (2\pi)^{2N} \sum_m \Phi\hat{\ }(m) \mid \mu\hat{\ }(m)|^2. \tag{3.7}$$

We designate the double integral on the left side of the equality in (3.7) by $I(\mu)$.

To establish the theorem, let $Z \subset T_N$ be a set closed in the torus topology and of ordinary capacity zero. By $B^T(x, \rho)$, we designate the open N-ball with center x and radius ρ in the torus topology, i.e., for x in T_N, $B^T(x, \rho) = \{y : y \text{ in } T_N \text{ and there exists } y' \text{ such that } y' \equiv y \bmod 2\pi \text{ in each variable and } |x - y'| < \rho\}$.

Next we let $\{B^T(x, \rho) : x \text{ in } Z\}$ be an open covering of Z in the torus topology. For each ρ with $0 < \rho < 1$, we extract a finite subcovering which contains the least number of balls $B^T(x, \rho)$. We define Z_ρ to be the closure in the torus topology of the union of the balls making the selected finite sub-covering.

For each ρ, with $0 < \rho < 1$, Z_ρ has positive Lebesgue measure and, therefore, positive ordinary capacity. Using the techniques given in the theorem in [4, p. 33], it follows from (1.3) and (1.4) and from (3.1) and (3.2) that there exists a nonnegative measure μ_ρ in $\mathscr{M}(T_N)$ of total mass one having its support in Z_ρ such that the equilibrium potential

$$U_\rho(x) = \int_{Z_\rho} \Phi(x - y)\, d\mu_\rho(y)$$

is a continuous periodic function in E_N taking a constant value on Z_ρ. Furthermore, it follows that this constant value is equal to $I(\mu_\rho)$ where

$$I(\mu_\rho) = \int_{Z_\rho} \int_{Z_\rho} \Phi(x - y)\, d\mu_\rho(x)\, d\mu_\rho(y). \tag{3.8}$$

Since each μ_ρ has total mass one, it follows from weak $*$ convergence that

$$\lim_{\rho \to 0} I(\mu_\rho) = +\infty. \tag{3.9}$$

For otherwise, there would exist a nonnegative measure μ in $\mathcal{M}(T_N)$ of total mass one having its support in Z and such that

$$\int_Z \int_Z \Phi(x - y)\, d\mu(x)\, d\mu(y) < +\infty,$$

which is clearly a contradiction to the fact that Z has ordinary capacity zero (see (3.2) and the definition given in Section 1).

Next, we introduce the real Hilbert space $\mathcal{O}\!\mathcal{L}(T_N)$. We say f is in $\mathcal{O}\!\mathcal{L}(T_N)$ if f is a real-valued function in $L^2(T_N)$ such that

$$\|f\|_\Phi^2 = \sum_m |f\,\hat{}(m)|^2\, |\Phi\,\hat{}(m)|^{-1} < +\infty. \tag{3.10}$$

Clearly $\mathcal{O}\!\mathcal{L}(T_N)$ is a real Hilbert space where the inner product $(f, g)_\Phi$ is given by

$$(f, g)_\Phi = \sum_m f\,\hat{}(m)\, g\,\hat{}(-m)\, |\Phi\,\hat{}(m)|^{-1}.$$

As mentioned above the equilibrium potential $U_\rho(x)$ which is defined for all x by the integral

$$U_\rho(x) = \int_{Z_\rho} \Phi(x - y)\, d\mu_\rho(y) \tag{3.11}$$

is such that

$$U_\rho(x) = I(\mu_\rho) \qquad \text{for } x \text{ in } Z_\rho\,, \tag{3.12}$$

where $I(\mu_\rho)$ is defined in (3.8).

Also, since μ_ρ has its support in Z_ρ, we see from (3.11) that

$$U_\rho\,\hat{}(m) = (2\pi)^N\, \Phi\,\hat{}(m)\, \mu_\rho\,\hat{}(m). \tag{3.13}$$

Consequently, it follows from (3.7), (3.8), (3.10), and (3.13) that

$$\| U_\rho \|_\Phi^2 = I(\mu_\rho). \tag{3.14}$$

Next, using (3.9) we select a sequence $\{\rho(j)\}_{j=1}^\infty$ such that

$$I(\mu_{\rho(j)})/j^4 \to +\infty \qquad \text{as} \quad j \to +\infty, \tag{3.15}$$

and we set

$$f_k(x) = \sum_{j=1}^k U_{\rho(j)}(x)/j^2[I(\mu_{\rho(j)})]^{1/2}. \tag{3.16}$$

Since $\mathfrak{A}(T_N)$ is a Hilbert space with respect to the norm $\|\ \|_\Phi$ given in (3.10), it follows from (3.14) and (3.16) that there is an f in $\mathfrak{A}(T_N)$ such that

$$\|f_k - f\|_\Phi \to 0 \qquad \text{as} \quad k \to \infty. \tag{3.17}$$

Also it follows from (3.3) and (3.10) that

$$\sum_m |f^\wedge(m)|^2 \, |m|^2 < +\infty. \tag{3.18}$$

To complete the proof of Theorem 2 we need only show

$$\int_0^1 |\partial f(x, t)/\partial t| \, dt = +\infty \qquad \text{for } x \text{ in } Z. \tag{3.19}$$

Since $|f(x, 1) - f(x, t)| \leqslant \int_t^1 |\partial f(x, s)/\partial s| \, ds$, (3.19) will follow once we show

$$\lim_{t \to 0} f(x, t) = +\infty \qquad \text{for } x \text{ in } Z. \tag{3.20}$$

In order to establish (3.20), we observe from (3.3), (3.10), and (3.17) that

$$\int_{T_N} |f_k(x) - f(x)|^2 \, dx \to 0 \qquad \text{as} \quad k \to +\infty. \tag{3.21}$$

Next, we set for $t > 0$

$$P(x, t) = \sum_m e^{i(m,x) - |m| t} \tag{3.22}$$

and observe from (2.7) that $P(x, t) = -G_t(x, t) + 1$. Consequently, it follows from (2.7) that for fixed positive t, $P(x, t)$ is a continuous periodic function of x. Furthermore, we have from (1.5) and (3.22) that

$$f_k(x, t) = (2\pi)^{-N} \int_{T_N} f_k(x - y) \, P(y, t) \, dy. \tag{3.23}$$

We conclude, consequently, from (3.21) and (3.23) that

$$\lim_{k \to \infty} f_k(x, t) = f(x, t) \qquad \text{for } x \text{ in } T_N \text{ and } t > 0. \tag{3.24}$$

Next, we recall that $\mu_{\rho(j)}$ is a nonnegative measure in $\mathscr{M}(T_N)$. Consequently, it follows from (3.1), (3.2), (3.11), and (3.16) that for each positive integer k

$$0 \leqslant f_k(x) \leqslant f_{k+1}(x) \qquad \text{for } x \text{ in } T_N. \tag{3.25}$$

From (2.7) and (3.22), we see that for $t > 0$, $P(x, t) > 0$ for x in T_N. We obtain, therefore, from (3.23) and (3.25) that for x in T_N and $t > 0$,

$$f_k(x, t) \leqslant f_{k+1}(x, t) \qquad \text{for} \quad k = 1, 2, \ldots . \tag{3.26}$$

From (3.24) and (3.26), we finally obtain

$$f(x, t) \geqslant f_k(x, t) \qquad \text{for } x \text{ in } T_N, \quad t > 0, \qquad \text{and} \qquad k = 1, 2, \ldots . \tag{3.27}$$

Next, recalling the definition of $Z_{\rho(j)}$, we see that if x^0 is in Z, there exists $B^T(x^0, r)$ with $r > 0$ such that $B^T(x^0, r) \subset Z_{\rho(j)}$. Also it follows from (3.12) that $U_{\rho(j)}$ takes the constant value $I(\mu_{\rho(j)})$ in $B^T(x^0, r)$. Consequently, we obtain from [7, p. 56] that $U_{\rho(j)}(x^0, t) \to I(\mu_{\rho(j)})$ as $t \to 0$. We conclude from (3.16) that

$$\lim_{t \to 0} f_k(x, t) = \sum_{j=1}^{k} [I(\mu_{\rho(j)})]^{1/2}/j^2 \qquad \text{for } x \text{ in } Z. \tag{3.28}$$

From (3.27) and (3.28), we next obtain that

$$\liminf_{t \to 0} f(x, t) \geqslant \sum_{j=1}^{k} [I(\mu_{\rho(j)})]^{1/2}/j^2 \qquad \text{for } x \text{ in } Z \text{ and } k = 1, 2, \ldots . \tag{3.29}$$

But (3.20) follows immediately from (3.15) and (3.29), and the proof of the theorem is complete.

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